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P-STRONGLY REGULAR NEAR-RINGS

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Abstract. In this paper we introduce the notion of P-strongly regular near-ring. We have shown that a zero-symmetric near-ring N is P-strongly regular if and only if N is P-regular and P is a completely semiprime ideal. We have also shown that in a P-strongly regular near-ring N, the following holds: (i) $Na + P$ is an ideal of N for any $a \in N$. (ii) Every P-prime ideal of N containing P is maximal. (iii) Every ideal I of N fulfills $I + P = I^2 + P$.

1. Introduction

Throughout this paper, N denotes a zero-symmetric right near-ring. A right N-subgroup (left N-subgroup) of N is a subgroup I of $(N, +)$ such that $IN \subseteq I (NI \subseteq I)$. A quasi-ideal of N is a subgroup Q of $(N, +)$ such that $QN \cap NQ \subseteq Q$. Right N-subgroups and left N-subgroups are quasi-ideals. The intersection of a family of quasi-ideals is again a quasi-ideal.

N is called regular, if for every element a of N there exists an element $x \in N$ such that $a = axa$. Let P be an ideal of N. Then the near-ring N is said to be a P-regular near-ring if for each $a \in N$, there exists an element $x \in N$ such that $a = axa + p$ for some $p \in P$. If $P = 0$, then a P-regular near-ring is a regular near-ring. Here the notion of P-regularity is a generalization of regularity. There are near-rings which are P-regular but not regular.

V. A. Andrunakievich [1] defined P-regular rings and S. J. Choi [3] extended the P-regularity of a ring to the P-regularity of a near-ring. In this paper we introduce the notion of P-strongly regular near-ring and obtain equivalent conditions for a near-ring to be P-strongly regular. We also introduce the notions of P-prime ideals and P-near-ring in this paper. I. Yakabe [7] characterized regular zero-symmetric near-rings without non-zero nilpotent elements in terms of quasi-ideals. In this paper we characterize P-strongly regular near-ring in terms of quasi-ideals. For the basic terminology and notation we refer to [6].

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2. Preliminaries

Definition 2.1. An ideal P of N is called completely semiprime if for any $a \in N$, $a^2 \in P$ implies $a \in P$.

Definition 2.2. An element $e \in N$ is called an P-idempotent if $e - e^2 \in P$.

For any non-empty subsets A, B of N, we write $\{n \in N \mid nB \subseteq A\}$ as $(A : B)$.

Lemma 2.3 ([6], Proposition 1.42)**.** *If A is an ideal and B is any subset of N, then $(A : B)$ is a left ideal of N.*

Lemma 2.4 ([2], Proposition 3.5)**.** *Let P be a completely semiprime ideal of N. Then $ab \in P$ implies $ba \in P$ and $aNb \subseteq P$ for any $a, b \in N$.*

Lemma 2.5. *If P is a completely semiprime ideal of N, then $(P : S)$ is an ideal of N for any non-empty subset S of N.*

Proof. By Lemma 2.3, $(P : S)$ is a left ideal of N. Let $x \in (P : S)$. Then $xS \subseteq P$ implies that for any $s \in S$, $xs \in P$. Thus $sx \in P$. Let $n \in N$. Now $(xns)^2 = xn(sx)ns \in P$. Since P is a completely semiprime ideal, $xns \in P$. Then $xnS \subseteq P$. Hence $(P : S)$ is an ideal of N. □

Lemma 2.6. *Let P be a completely semiprime ideal of N. If $a \in N$ is an P-idempotent, then for any $n \in N$, $an = ana + p$ for some $p \in P$.*

Proof. Let $a \in N$ be an P-idempotent. Then $a^2 = a + p_1$ for some $p_1 \in P$. Let $n \in N$. Now $(an - ana)a = ana - (an(a + p_1) - ana + ana) = p_2$ for some $p_2 \in P$. By Lemma 2.4, $an(an - ana) \in P$ and $ana(an - ana) \in P$. Thus $(an - ana)^2 \in P$ implies that $an - ana \in P$. Hence $an = ana + p$ for some $p \in P$. □

3. *P*-strongly regular

Definition 3.1. A near-ring N is said to be strongly regular if for each $a \in N$, there exists an element $x \in N$ such that $a = xa^2$.

Now we introduce P-strongly regular near-ring.

Definition 3.2. A near-ring N is said to be P-strongly regular if for each $a \in N$, there exists an element $x \in N$ such that $a = xa^2 + p$ for some $p \in P$.

If $P = 0$, then a P-strongly regular near-ring is a strongly regular near-ring. If N is strongly regular, then N is P-strongly regular for all ideals P of N. But P-strongly regular near-ring for any ideal P of N need not be strongly regular near-ring as the following example shows.

Example 3.3. Let $N = \{0, a, b, c\}$ be the Klein's four group. Define multiplication in N as follows:

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	0	0	a
b	0	a	b	b
c	0	a	b	c

Then $(N, +, \cdot)$ is a near-ring (see Pilz [6], p. 407, scheme 8). Here the ideals are $\{0\}$, $\{0, a\}$ and N. Let $P = \{0, a\}$. Clearly N is P-strongly regular but not strongly regular since $a \notin Na^2$.

Theorem 3.4. *N is P-strongly regular if and only if N is P-regular and P is a completely semiprime ideal.*

Proof. Assume that N is P-strongly regular. Suppose that $a \in N$ such that $a^2 \in P$. Since N is P-strongly regular, there exists $x \in N$ such that $a = xa^2+p_1$ for some $p_1 \in P$. Then $a \in P$. Thus P is a completely semiprime ideal. Let $a \in N$ such that $a = xa^2 + p$ for some $p \in P$. Now $(a - axa)a = a^2 - (a(a-p) - a^2 + a^2) = p_2$ for some $p_2 \in P$. By Lemma 2.4, $a(a - axa) \in P$ and $axa(a - axa) \in P$. Then $(a - axa)^2 \in P$ implies that $a - axa \in P$. Thus $a = axa + p_3$ for some $p_3 \in P$ and hence N is P-regular. Conversely, assume that N is P-regular and P is a completely semiprime ideal. Let $a \in N$ be such that $a = axa + p$ for some $x \in N$ and $p \in P$. Thus xa is an P-idempotent. Now $a = (axa + p)xa + p = a(xax)a + p^{'}$ for some $p^{'} \in P$. By Lemma 2.6, $a = a(xaxxa+p^{''})a+p^{'} = a(xax^2a^2+p_1)+p^{'} = a(xax^2a^2+p_1) - axax^2a^2 + axax^2a^2 + p^{'} = axax^2a^2 + p^{'''} = ya^2 + p^{'''}$ for some $p_1, p^{''}, p^{'''} \in P$ and $y = axax^2 \in N$. Hence N is P-strongly regular. □

Definition 3.5. An ideal A of N is said to be prime if $BC \subseteq A$ implies $B \subseteq A$ or $C \subseteq A$ for ideals B, C of N.

Definition 3.6. An ideal A of N is said to be P-prime if $BC + P \subseteq A$ implies $B \subseteq A$ or $C \subseteq A$ for ideals B, C of N.

If A is a prime ideal, then clearly A is a P-prime ideal for any ideal P. Now we give an example of a P-prime ideal but not prime.

Example 3.7. Let $N = \{0, a, b, c\}$ be the Klein's four group. Define multiplication in N as follows:

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
c	0	a	b	c

Then $(N, +, \cdot)$ is a near-ring (see Pilz [6], p. 407, scheme 7). Here the ideals are $\{0\}$, $\{0, a\}$, $\{0, b\}$ and N. Let $P = \{0, b\}$. Clearly $\{0\}$ is P-prime but not prime since $\{0, a\}\{0, b\} \subseteq \{0\}$ but $\{0, a\} \nsubseteq \{0\}$ and $\{0, b\} \nsubseteq \{0\}$.

Theorem 3.8. *Let N be a P-strongly regular near-ring. Then*

(1) *$Na + P$ is an ideal of N for any $a \in N$.*

(2) *Every P-prime ideal of N containing P is maximal.*

(3) *Every ideal I of N fulfills $I + P = I^2 + P$.*

Proof. (1) Assume that N is a P-strongly regular near-ring. By Theorem 3.4, N is P-regular and P is a completely semiprime ideal. Let $a \in N$. Now $a = axa + p_1$ for some $x \in N$ and $p_1 \in P$. Then xa is an P-idempotent. Now for any $n \in N$, $na = n(axa + p_1) - naxa + naxa = naxa + p_2$ for some $p_2 \in P$ implies that $na \in Nxa + P$. Thus $Na + P \subseteq Nxa + P$. Clearly $Nxa+P \subseteq Na+P$. Therefore $Na+P = Nxa+P$. Let $S = \{n-nxa \mid n \in N\}$. Now for any $n \in N$, $nxa = nx(axa + p_1) - nxaxa + nxaxa = nxaxa + p_3$ for some $p_3 \in P$. Thus $(n - nxa)Nxa \subseteq P$ implies that $Nxa(n - nxa) \subseteq P$. Therefore $Nxa + P \subseteq (P : S)$. Let $y \in (P : S)$. Then $yS \subseteq P$. Thus $y(y - yxa) \in P$. Since P is completely semiprime, $(y - yxa)y \in P$. Therefore $y^2 = yxay + p$ for some $p \in P$. Since N is P-strongly regular, there exists $z \in N$ such that $y = zy^2 + p^{'}$ for some $p^{'} \in P$. Then $zy^2 = y + p^{''}$ for some $p^{''} \in P$. Now $zy^2 = z(yxay + p) - zyxay + zyxay = zy(xay) + p_1$ for some $p_1 \in P$. By Lemma 2.6, $zy^2 = zy(xayxa + p_2) + p_1$ for some $p_2 \in P$. Thus $zy^2 = zyxayxa + p_3$ for some $p_3 \in P$. Then $y \in Nxa + P$ implies that $(P : S) \subseteq Nxa + P$. Hence $(P : S) = Nxa + P = Na + P$. By Lemma 2.5, $Na + P$ is an ideal of N.

(2) Let A be a P-prime ideal of N containing P and suppose $A \subset M$ for an ideal M of N. Let $b \in M \backslash A$. Now $b = xb^2 + p_1$ for some $x \in N$ and $p_1 \in P$. Let $n \in N$. Now $nb = n(xb^2 + p_1) - nxb^2 + nxb^2 = nxb^2 + p_2$ for some $p_2 \in P$. Then $(n - nxb)b \in P$. By Lemma 2.4, $N(n - nxb)Nb \subseteq P$. Thus $N(n - nxb)Nb + P \subseteq A$ implies that $[(N(n - nxb) + P)(Nb + P)] + P \subseteq A$. Since A is a P-prime ideal, $N(n - nxb) \subseteq A$ or $Nb \subseteq A$. Suppose $Nb \subseteq A$. Since $b = xb^2 + p_1 \in Nb + P$, we have $b \in A$, a contradiction. Suppose $N(n - nxb) \subseteq A$. Then $n - nxb \in A \subset M$. Since $b \in M$, $nxb \in M$. Then $n \in M$. Thus $M = N$. Hence A is maximal.

(3) Let I be an ideal of N containing P. Clearly $I^2 + P \subseteq I + P$. Let $a \in I + P$. Since N is P-strongly regular, we have $a = xa^2 + p$ for some $x \in N$ and $p \in P$. Then $a = (xa)a + p \in I^2 + P$. Hence $I + P = I^2 + P$. □

Corollary 3.9 ([4], Theorem 5)**.** *Let N be a strongly regular near-ring. Then*

(1) *Every N-subgroup of N is an ideal.*

(2) *Every prime ideal of N is maximal.*

(3) *Every ideal I of N fulfills $I = I^2$.*

I. Yakabe [7] proved that if a near-ring N is regular, then every quasi-ideal Q of N has the form $QNQ = Q$. It can be generalized in the case of a P-strongly regular near-ring.

Lemma 3.10 ([3], Theorem 2.6)**.** *If N is a P-regular near-ring, then every quasi-ideal Q of N has the form $Q + P = QNQ + P$.*

Definition 3.11. A near-ring N is said to be an S-near-ring, if $a \in Na$ for every $a \in N$.

Definition 3.12. A near-ring N is said to be a P-near-ring, if $a \in Na + P$ for every $a \in N$.

Clearly every S-near-ring is a P-near-ring for any ideal P.

Theorem 3.13. *The following conditions are equivalent*:

(1) *N is P-strongly regular.*
(2) *N is a P-near-ring and for every quasi-ideal Q, $QN + P = Q + P = Q^2 + P$.*
(3) *N is a P-near-ring and for any two left N-subgroups L_1,L_2 of N, $(L_1 + P) \cap (L_2 + P) = L_1L_2 + P$.*

Proof. (1) ⇒ (2) Clearly N is a P-near-ring. Let Q be a quasi-ideal of N. Any element x of $QN + P$ has the form $x = qn + p_1$ for some $p_1 \in P$, $q \in Q$ and $n \in N$. Then $x = (qyq + p_2)n + p_1 = q(yqn) + p_3$ for some $p_2, p_3 \in P$ and $y \in N$. By Lemma 2.6, $x = q(yqnyq + p_4) + p_3 = qyqnyq + p_5$ for some $p_4, p_5 \in P$. Therefore $QN + P \subseteq QNQ + P$. By Lemma 3.10, $Q + P = QNQ + P \subseteq QN + P \subseteq QNQ + P$. Now $Q^2 + P \subseteq QN + P = Q + P$. Let $q_1 \in Q$ and $p_1 \in P$. Now $q_1 + p_1 = q_2nq_3 + p_2 = (q_4 + p_3)q_3 + p_2 = q_4q_3 + p_4$ for some $p_2, p_3, p_4 \in P$, $q_2, q_3, q_4 \in Q$ and $n \in N$. Thus $Q + P \subseteq Q^2 + P$. Hence $QN + P = Q + P = Q^2 + P$.

(2) ⇒ (3) Let L_1,L_2 be left N-subgroups of N. Now $L_1L_2 + P \subseteq (L_1 + P) \cap (L_2 + P) \subseteq ((L_1 + P) \cap (L_2 + P)) + P = ((L_1 + P) \cap (L_2 + P))^2 + P \subseteq (L_1 + P)(L_2 + P) + P \subseteq L_1L_2 + P$. Hence $(L_1 + P) \cap (L_2 + P) = L_1L_2 + P$.

(3) ⇒ (1) Let $a \in N$. Since Na and N are left N-subgroups of N, we have $Na+P = NaNa+P$ and $Na+P = NaN+P$. So we get $Na+P = NaNa+P = Na^2 + P$. Since N is a P-near-ring, we have $a \in Na + P = Na^2 + P$. Hence N is P-strongly regular. □

Corollary 3.14 ([7], Theorem 1)**.** *The following conditions on a zero-symmetric near-ring N are equivalent*:

(1) *N is regular and has no non-zero nilpotent elements.*
(2) *N is an S-near-ring and every quasi-ideal of N is an idempotent right N-subgroup of N.*
(3) *N is an S-near-ring and for any two left N-subgroups L_1,L_2 of N, $L_1 \cap L_2 = L_1L_2$.*

Proof. If N is regular and has no non-zero nilpotent elements, then N is P-strongly regular. □

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